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Shakedown limits for reinforced beam structures under fluctuating loads

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Abstract

Shakedown of planar beam structures, with possible different reinforcements in the upper and lower layers of the beam members, subjected to quasi-statically as well as dynamically fluctuating loads is studied. With an assumption of elastic perfectly-plastic behaviour of the beams in bending, a reduced kinematic formulation for the safety factor determining shakedown limits is constructed, which is proved to be equivalent to (but simpler than) the original one, and ready for use in solving practical problems.

1. Introduction

Consider a planar frame of n beams of possible variable bending stiffness associated naturally with a system of axial coordinates $\{0 \leqslant x_i \leqslant l_i\}_{i=1}^{n}$. Let $\dot{\kappa}_i(x_i, t)$ denote the plastic curvature rate (t is the time parameter); the fictitious elastic moment response of the structure to external agencies $M_i^e(x_i, t)$ in assumption of its perfectly elastic behaviour is confined to a certain time-independent loading domain $\mathscr{L}$, the shakedown boundary of which should be determined (Pham 1996):

$$\mathscr{L}: M_i^-(x_i) \leqslant M_i^e(x_i, t) \leqslant M_i^+(x_i), \quad i = 1, \ldots, n. \tag{1}$$

Usually in reinforced concrete beams, the reinforcement is placed in both the lower and upper layers, and generally—in different amounts. If the reinforcement is not identical in both layers, the corresponding yield moments will also differ. Yield moments for simple bending are denoted by $M_{Yi}^+(x_i)$ for positive bending (tension of the lower layer) and $M_{Yi}^-(x_i)$ for negative bending (tension of the upper layer); the simplified physical assumption of elastic perfectly-plastic behaviour of the beams in bending is adopted (Save and Massonnnet, 1972; Lubliner, 1990). The dissipation function $D(\dot{\kappa}_i)$ should take the form

$$D(\dot{\kappa}) = \begin{cases} M_{Yi}^+\dot{\kappa}_i, & \dot{\kappa}_i \geqslant 0 \\ -M_{Yi}^-\dot{\kappa}_i, & \dot{\kappa}_i < 0 \end{cases}. \tag{2}$$

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From eqn (2) one can see that

$$D(\Lambda_i \cdot \Delta\kappa_i) = \begin{cases} \Lambda_i \cdot D(\Delta\kappa_i), & \Lambda_i \geqslant 0 \\ |\Lambda_i| \cdot D(-\Delta\kappa_i), & \Lambda_i < 0 \end{cases}. \tag{3}$$

Shakedown kinematic theorem (Koiter, 1963; König, 1987; Pham 1996) applied to our beam structures can be expressed through the shakedown factor k_s in generalized variables (moment and curvature) as

$$k_s^{-1} = \sup_{M_i^e \in \mathscr{L}; \dot{\kappa} \in \mathscr{A}} \frac{\sum_{i=1}^{n} \int_0^{l_i} \mathrm{d}x_i \int_0^T M_i^e \dot{\kappa}_i \, \mathrm{d}t}{\sum_{i=1}^{n} \int_0^{l_i} \mathrm{d}x_i \int_0^T D(\dot{\kappa}_i) \, \mathrm{d}t} \tag{4}$$

(at $k_s > 1$ the structure will shake down, while it will not at $k_s < 1$), where the plastic curvature rate $\dot{\kappa}_i(x_i, t)$ belongs to the set of compatible plastic curvature cycles:

$$\mathscr{A} = \left\{ \dot{\kappa}_i \in \mathscr{K} \mid \Delta\kappa_i = \int_0^T \dot{\kappa}_i \, \mathrm{d}t \in \mathscr{C} \right\}, \tag{5}$$

$\mathscr{K}$ is the set of plastic curvature fields; $\mathscr{C} \subset \mathscr{K}$ is the subset from those fields that satisfy compatibility conditions (kinematic constraints for a particular problem); $\Delta\kappa_i$ is called the (compatible) curvature increment over a cycle.

The solution of the problem (4), (5) for a structure under quasi-static as well as dynamic loading should yield the shakedown limits for the particular problem considered.

2. A Reduced formulation

The formulation (3), (4) is hard to be used, so our objective is to simplify it. Any plastic curvature rate field $\dot{\kappa}_i \in \mathscr{A}$ can be decomposed as

$$\dot{\kappa}_i(x_i, t) = \Lambda_i(x_i, t)[\Delta\kappa_i(x_i) + \kappa_i^0(x_i)], \quad \Delta\kappa_i \in \mathscr{C}, \quad \kappa_i^0 \in \mathscr{K},$$

$$\kappa_i^0(x_i) = 0 \quad \text{if } x_i \in L_i = \{0 \leqslant x_i \leqslant l_i \mid \Delta\kappa_i(x_i) \neq 0\}$$

$$\text{(in addition, define } L_{0i} = \{0 \leqslant x_i \leqslant l_i \mid \Delta\kappa_i(x_i) = 0\}); \tag{6}$$

$$\int_0^T \Lambda_i \, \mathrm{d}t = 1, \quad \int_0^T (|\Lambda_i| - \Lambda_i) \, \mathrm{d}t = S_i(x_i), \quad x_i \in L_i;$$

$$\int_0^T \Lambda_i \, \mathrm{d}t = 0, \quad \int_0^T |\Lambda_i| \, \mathrm{d}t = 1, \quad x_i \in L_{0i}; \tag{7}$$

$\Delta\kappa_i(x_i)$, $\kappa^{0i}(x_i)$ and $\Lambda_i(x_i, t)$ otherwise are arbitrary functions; $S_i(x_i)/2$ measures the absolute value of the integral of $\Lambda_i(x_i, t)$ over the time intervals of the cycle, during which $\Lambda_i(x_i, t)$ is negative. Clearly $S_i(x_i)$ can take all possible values.

One can see that the decomposition (6), (7) is generally enough to cover all possible plastic fields for the reinforced beams in bending (for general three-dimensional structures we would not have such a "relatively simple" decomposition: the plastic strain rate field may take not only opposite directions over times at different points differently as indicated here, but can also rotate in the strain space).

Equation (6) indicates that, generally, as different points in a structure can follow different deformation patterns, the plastic curvature field would not appear globally in pure monotonous incremental ($\dot{\kappa} \geqslant 0, \Delta\kappa_i \neq 0$) or alternating ($\dot{\kappa}_i \neq 0, \Delta\kappa_i \equiv 0$) modes. Luckily enough, we would get a kind of separation in the final form of reduced kinematic theorem determining the shakedown factor by a non trivial equivalent transformation process that followed.

With (7) one can verify

$$\int_0^T (|\Lambda_i| + \Lambda_i)\,\mathrm{d}t = S_i + 2, \quad \int_0^T |\Lambda_i|\,\mathrm{d}t = S_i + 1, \quad x_i \in L_i,$$

$$\int_0^T (|\Lambda_i| + \Lambda_i)\,\mathrm{d}t = 1, \quad \int_0^T (|\Lambda_i| - \Lambda_i)\,\mathrm{d}t = 1, \quad x_i \in V_{0i}. \tag{8}$$

Denote

$$\max_t M_i^e(x_i, t)(\Delta\kappa_i + \kappa_i^0) = M_i^e(x_i, t_{xi}^u)(\Delta\kappa_i + \kappa_i^0) = M_i^u(x_i)(\Delta\kappa_i + \kappa_i^0),$$

$$\min_t M_i^e(x_i, t)(\Delta\kappa_i + \kappa_i^0) = M_i^e(x_i, t_{xi}^l)(\Delta\kappa_i + \kappa_i^0) = M_i^l(x_i)(\Delta\kappa_i + \kappa_i^0), \tag{9}$$

t_{xi}^u and t_{xi}^l denote the time instants, at which the corresponding maximums and minimums are reached. Clearly

$$M_i^u(x_i) = M_i^+(x_i), \quad M_i^l(x_i) = M_i^-(x_i) \quad \text{if } \Delta\kappa_i + \kappa_i^0 \geqslant 0,$$

$$M_i^u(x_i) = M_i^-(x_i), \quad M_i^l(x_i) = M_i^+(x_i) \quad \text{if } \Delta\kappa_i + \kappa_i^0 < 0. \tag{10}$$

Substituting (6) into (4) we obtain [from now on $\Delta\kappa_i, \kappa_i^0$, Λ_i are understood implicitly to satisfy (6) and (7), while $M_i^e \in \mathscr{L}$]

$$k_s^{-1} = \sup_{\Delta\kappa_i, \kappa_i^0, \Lambda_i} \frac{\sum_{i=1}^n \int_0^{l_i} \mathrm{d}x_i \int_0^T M_i^e(\Delta\kappa_i + \kappa_i^0)\Lambda_i\,\mathrm{d}t}{\sum_{i=1}^n \int_0^{l_i} \mathrm{d}x_i \int_0^T D(\Lambda_i(\Delta\kappa_i + \kappa_i^0))\,\mathrm{d}t}. \tag{11}$$

With (3), (7)–(9) on can verify that

$$\int_0^T M_i^e(\Delta\kappa_i+\kappa_i^0)\Lambda_i\,\mathrm{d}t = \int_0^T M_i^e(\Delta\kappa_i+\kappa_i^0)\left(\frac{|\Lambda_i|+\Lambda_i}{2}-\frac{|\Lambda_i|-\Lambda_i}{2}\right)\mathrm{d}t$$

$$\leqslant M_i^u\left[\left(\frac{S_i}{2}+1\right)\Delta\kappa_i+\frac{1}{2}\kappa_i^0\right]-M_i^l\left[\frac{S_i}{2}\Delta\kappa_i+\frac{1}{2}\kappa_i^0\right], \qquad (12)$$

$$\int_0^T D(\Lambda_i(\Delta\kappa_i-\kappa_i^0))\,\mathrm{d}t = \int_0^T D\left(\left(\frac{|\Lambda_i|+\Lambda_i}{2}-\frac{|\Lambda_i|-\Lambda_i}{2}\right)(\Delta\kappa_i+\kappa_i^0)\right)\mathrm{d}t$$

$$=\left(\frac{S_i}{2}+1\right)D(\Delta\kappa_i)+\frac{S_i}{2}D(-\Delta\kappa_i)+\frac{1}{2}D(\kappa_i^0)+\frac{1}{2}D(-\kappa_i^0). \qquad (13)$$

From (11)–(13) follows

$$k_s^{-1}\leqslant \sup_{\Delta\kappa_i,\kappa_i^0,S_i\geqslant 0}\frac{\sum_{i=1}^{n}\int_0^{l_i}\left\{M_i^u\left[\left(\frac{S_i}{2}+1\right)\Delta\kappa_i+\frac{1}{2}\kappa_i^0\right]-M_i^l\left[\frac{S_i}{2}\Delta\kappa_i+\frac{1}{2}\kappa_i^0\right]\right\}\mathrm{d}x_i}{\sum_{i=1}^{n}\int_0^{l_i}\left[\left(\frac{S_i}{2}+1\right)D(\Delta\kappa_i)+\frac{S_i}{2}D(-\Delta\kappa_i)+\frac{1}{2}D(\kappa_i^0)+\frac{1}{2}D(-\kappa_i^0)\right]\mathrm{d}x_i}. \qquad (14)$$

To verify that the expression in the right hand side of (14) can be reached by that of (11) with an appropriately chosen trial field and then the inequality sign in (14) can be suppressed, we take a trial field $\Lambda_i(x_i, t)$ satisfying (7) (therefore the field is an admissible)

$$\Lambda_i(x_i,t)=\begin{cases}\frac{1}{2}(S_i+s)\delta(t-t_{xi}^u)-\frac{1}{2}S_i\delta(t-t_{xi}^l), & x_i\in L_i\\ \frac{1}{2}\delta(t-t_{xi}^u)-\frac{1}{2}\delta(t-t_{xi}^l), & x_i\in L_{0i}\end{cases} \qquad (15)$$

[$\delta(t)$ is the Dirac function] and substitute it into (11):

$$\int_0^{l_i}\mathrm{d}x_i\int_0^T M_i^e(\Delta\kappa_i+\kappa_i^0)\Lambda_i\,\mathrm{d}t=\int_{L_i}\mathrm{d}x_i\int_0^T M_i^e\Delta\kappa_i\Lambda_i\,\mathrm{d}t+\int_{L_{0i}}\mathrm{d}x_i\int_0^T M_i^e\kappa_i^0\Lambda_i\,\mathrm{d}t$$

$$=\int_{L_i}\left[M_i^u\left(\frac{S_i}{2}+1\right)\Delta\kappa_i-M_i^l\frac{S_i}{2}\Delta\kappa_i\right]\mathrm{d}x_i+\int_{L_{0i}}\left[M_i^u\frac{1}{2}\kappa_i^0-M_i^l\frac{1}{2}\kappa_i^0\right]\mathrm{d}x_i$$

$$=\int_0^{l_i}\left\{M_i^u\left[\left(\frac{S_i}{2}+1\right)\Delta\kappa_i+\frac{1}{2}\kappa_i^0\right]-M_i^l\left[\frac{S_i}{2}\Delta\kappa_i+\frac{1}{2}\kappa_i^0\right]\right\}\mathrm{d}x_i,$$

$$\int_0^{l_i}\mathrm{d}x_i\int_0^T D(\Lambda_i(\Delta\kappa_i+\kappa_i^0))\,\mathrm{d}t=\int_{L_i}\mathrm{d}x_i\int_0^T D(\Lambda_i\Delta\kappa_i)\,\mathrm{d}t+\int_{L_{0i}}\mathrm{d}x_i\int_0^T D(\Lambda_i\kappa_i^0)\,\mathrm{d}t$$

$$=\int_0^{l_i}\left[\left(\frac{S_i}{2}+1\right)D(\Delta\kappa_i)+\frac{S_i}{2}D(-\Delta\kappa_i)+\frac{1}{2}D(\kappa_i^0)+\frac{1}{2}D(-\kappa_i^0)\right]\mathrm{d}x_i,$$

so

$$k_s^{-1} = \sup_{\Delta\kappa_i,\kappa_i^0,\Lambda_i} \frac{\sum_{i=1}^{n} \int_0^{l_i} \mathrm{d}x_i \int_0^T M_i^e(\Delta\kappa_i + \kappa_i^0)\Lambda_i \,\mathrm{d}t}{\sum_{i=1}^{n} \int_0^{l_i} \mathrm{d}x_i \int_0^T D(\Lambda_i(\Delta\kappa_i + \kappa_i^0)) \,\mathrm{d}t}$$

$$\geqslant \sup_{\Delta\kappa_i,\kappa_i^0,S_i \geqslant 0} \frac{\sum_{i=1}^{n} \int_0^{l_i} \left\{ M_i^u \left[\left(\frac{S_i}{2} + 1 \right) \Delta\kappa_i + \frac{1}{2}\kappa_i^0 \right] - M_i^l \left[\frac{S_i}{2}\Delta\kappa_i + \frac{1}{2}\kappa_i^0 \right] \right\} \mathrm{d}x_i}{\sum_{i=1}^{n} \int_0^{l_i} \left[\left(\frac{S_i}{2} + 1 \right) D(\Delta\kappa_i) + \frac{S_i}{2} D(-\Delta\kappa_i) + \frac{1}{2} D(\kappa_i^0) + \frac{1}{2} D(-\kappa_i^0) \right] \mathrm{d}x_i}$$

(the supremum over Λ_i should be greater than the expression obtained from an admissible one).

The last inequality together with (14) implies

$$k_s^{-1} = \sup_{\Delta\kappa_i,\kappa_i^0,S_i \geqslant 0} \frac{\sum_{i=1}^{n} \int_0^{l_i} \left\{ M_i^u \left[\left(\frac{S_i}{2} + 1 \right) \Delta\kappa_i + \frac{1}{2}\kappa_i^0 \right] - M_i^l \left[\frac{S_i}{2}\Delta\kappa_i + \frac{1}{2}\kappa_i^0 \right] \right\} \mathrm{d}x_i}{\sum_{i=1}^{n} \int_0^{l_i} \left[\left(\frac{s_i}{2} + 1 \right) D(\Delta\kappa_i) + \frac{S_i}{2} D(-\Delta\kappa_i) + \frac{1}{2} D(\kappa_i^0) + \frac{1}{2} D(-\kappa_i^0) \right] \mathrm{d}x_i}. \tag{16}$$

Introduce a new function $\bar{\kappa}_i(x_i)$ that

$$\bar{\kappa}_i(x_i) = 0, \quad x_i \in L_i,$$
$$\bar{\kappa}_i^0(x_i) = S_i(x_i)\bar{\kappa}_i(x_i), \quad S_i(x_i) \geqslant 0, \quad x_i \in L_{0i} \tag{17}$$

[functions $S_i(x_i)$ for $x_i \in L_i$ has already been defined in (7)—clearly $S_i(x_i)$ can be an arbitrary function], then (16) can be rewritten as

$$k_s^{-1} = \sup_{\Delta\kappa_i,\kappa_i^0,S_i \geqslant 0} \frac{\sum_{i=1}^{n} \int_0^{l_i} \left[M_i^u \Delta\kappa_i + \frac{1}{2} S_i (M_i^u - M_i^l)(\Delta\kappa_i + \bar{\kappa}_i) \right] \mathrm{d}x_i}{\sum_{i=1}^{n} \int_0^{l_i} \left[D(\Delta\kappa_i) + \frac{S_i}{2} (D(\Delta\kappa_i + \bar{\kappa}_i) + D(-\Delta\kappa_i - \bar{\kappa}_i) \right] \mathrm{d}x_i}. \tag{18}$$

Define

$$\bar{S}_i(x_i) = \frac{1}{2} S_i(x_i)[D(\Delta\kappa_i + \bar{\kappa}_i) + D(-\Delta\kappa_i - \bar{\kappa}_i)], \quad X = \sum_{i=1}^{n} \int_0^{l_i} \bar{S}_i(x_i) \,\mathrm{d}x_i \tag{19}$$

[as $S_i(x_i)$ is an arbitrary positive function, X is also an arbitrary positive scalar],

$$U = \max_{1 \leqslant i \leqslant n} \max_{0 \leqslant x_i \leqslant l_i} \frac{[M_i^u(x_i) - M_i^l(x_i)][\Delta\kappa_i(x_i) + \bar{\kappa}_i(x_i)]}{D(\Delta\kappa_i + \bar{\kappa}_i) + D(-\Delta\kappa_i - \bar{\kappa}_i)} \tag{20}$$

with x_j^u being the point where the maximum is reached.

Substituting (19) into (18) and taking into account (20), one deduces

$$k_s^{-1} = \sup_{\Delta\kappa_i,\bar{\kappa}_i,\bar{S}_i \geqslant 0} \frac{\sum_{i=1}^{n} \int_0^{l_i} \left[M_i^u \Delta\kappa_i + \bar{S}_i \frac{(M_i^u - M_i^l)(\Delta\kappa_i + \bar{\kappa}_i)}{D(\Delta\kappa_i + \bar{\kappa}_i) + D(-\Delta\kappa_i - \bar{\kappa}_i)} \right] \mathrm{d}x_i}{\sum_{i=1}^{n} \int_0^{l_i} [D(\Delta\kappa_i) + \bar{S}_i] \, \mathrm{d}x_i}$$

$$\leqslant \sup_{\Delta\kappa_i,\bar{\kappa}_i,X \geqslant 0} \frac{\sum_{i=1}^{n} \int_0^{l_i} M_i^u \Delta\kappa_i \, \mathrm{d}x + X \cdot U}{\sum_{i=1}^{n} \int_0^{l_i} D(\Delta\kappa_i) \, \mathrm{d}x_i + X}. \quad (21)$$

On the other hand, putting an admissible variable

$$\bar{S}_i(x_i) = X \cdot \delta(x_i - x_j^u)[x_j^u \text{ is the maximum point of (20)}] \quad (22)$$

into the right hand of the equality in (21), we get the exact expression after the inequality sign [the procedure is similar to that from (14)–(16)]. Thus, the expression is reachable and the inequality can be changed for the equality, that is

$$k_s^{-1} = \sup_{\Delta\kappa_i,\bar{\kappa}_i,X \geqslant 0} \frac{\sum_{i=1}^{n} \int_0^{l_i} M_i^u \Delta\kappa_i \, \mathrm{d}x_i + X \cdot U}{\sum_{i=1}^{n} \int_0^{l_i} D(\Delta\kappa_i) \, \mathrm{d}x_i + X}. \quad (23)$$

The expression after sup in (23) depends monotonically upon $X \in [0, +\infty)$, therefore, the supremum over $X \geqslant 0$ is attained at $X = 0$ or $X = +\infty$. Hence

$$k_s^{-1} = \max\{I, A\}, \quad (24)$$

where

$$I = \sup_{\Delta\kappa_i \in \mathscr{C}} \frac{\sum_{i=1}^{n} \int_0^{l_i} M_i^u \Delta\kappa_i \, \mathrm{d}x_i}{\sum_{i=1}^{n} \int_0^{l_i} D(\Delta\kappa_i) \, \mathrm{d}x_i}$$

$$= \sup_{\Delta\kappa_i \in \mathscr{C}} \frac{\sum_{i=1}^{n} \int_0^{l_i} \max\{M_i^+ \Delta\kappa_i, M_i^- \Delta\kappa_i\} \, \mathrm{d}x_i}{\sum_{i=1}^{n} \int_0^{l_i} D(\Delta\kappa_i) \, \mathrm{d}x_i}, \quad (25)$$

$$A = \sup_{\Delta\kappa_i \in \mathscr{C}, \bar{\kappa}_i \in \mathscr{K}} U$$

$$= \sup_{\Delta\kappa_i \in \mathscr{C}, \bar{\kappa}_i \in \mathscr{K}, 0 \leqslant x_i \leqslant l_i} \frac{[M_i^u(x_i) - M_i^l(x_i)][\Delta\kappa_i(x_i) + \bar{\kappa}_i(x_i)]}{D(\Delta\kappa_i + \bar{\kappa}_i) + D(-\Delta\kappa_i - \bar{\kappa}_i)}$$

$$= \sup_{i, 0 \leqslant x_i \leqslant l_i} \frac{M_i^+(x_i) - M_i^-(x_i)}{M_{Yi}^+ + M_{Yi}^-}. \tag{26}$$

Equation (25) represents the incremental collapse mode (with $\Delta\kappa_i \in \mathscr{C}$ being a compatible plastic increment over a cycle), while (26) reflects the alternating plasticity collapse mode.

Thus the original difficult problem (4), (5) has been transformed into a simpler reduced form (24)–(26). The reduced form does not contain time integrals and is separated into the separated terms I, representing the incremental collapse, and A, representing the alternating plasticity mode. It is equivalent to the original formulation (4), (5) under physical assumption (2), and applies to general dynamic loading processes, not just to the quasistatic ones, to which the plastic limit and also the classical shakedown analysis are usually restricted to. We have given it a rigorous proof without any restrictions. So one is right to use eqns (24)–(26) directly in applications. The respective plastic limit problem can be considered as a limit case of the shakedown one (in case of static loading), with the plastic limit factor k_p being given as

$$k_p^{-1} = \sup_{\kappa_i \in \mathscr{C}} \frac{\sum_{i=1}^{n} \int_0^{l_i} M_i^e(x_i)\dot{\kappa}_i \, \mathrm{d}x_i}{\sum_{i=1}^{n} \int_0^{l_i} D(\dot{\kappa}_i) \, \mathrm{d}x_i}, \tag{27}$$

where $\dot{\kappa}_i$ is the collapse curvature rate field, which should be a compatible strain field: $M_i^e(x_i)$—the fictitious elastic moment distribution corresponding to the collapse point.

Certain similarity between (24)–(25) and (27) indicates that the methods available in solving the latter (Symonds and Neal, 1951; Hodge, 1959; Save and Massonnet, 1972; Lubliner, 1990) can be developed for use in solving the former. In shakedown analysis the boundary $M_i^+(x_i)M_i^-(x_i)$ [see (1)] of the elastic moment response of the structure to external agencies (quasi-static as well as dynamic) should be determined a priori. Then the solution of (26) is simple and straightforward. (25) presents certain difficulties as it requires the solution of a nonlinear optimization problem over compatible fields $\Delta\kappa_i \in \mathscr{C}$. Plastic incremental collapse mechanisms can be constructed for use there. We will see some simple illustrations in the next section.

3. Examples

Consider a uniform beam $0 \leqslant x \leqslant L$ clamped at one end and simply supported at the other under quasi-static uniform transverse loads

$$q_0^- \leqslant q(t) \leqslant q_0^+. \tag{28}$$

The fictitious elastic moment response of the beam to external loads (28) should have the form

$$M^e(x,t) = -q\left(\frac{x^2}{2} - \frac{5Lx}{8} + \frac{L^2}{8}\right), \tag{29}$$

which is confined to the boundary limits

$$M^+(x) = \max_t M^e(x,t) = \begin{cases} -q_0^-\left(\dfrac{x^2}{2} - \dfrac{5Lx}{8} + \dfrac{L^2}{8}\right), & 0 \leqslant x \leqslant L/4 \\ -q_0^+\left(\dfrac{x^2}{2} - \dfrac{5Lx}{8} + \dfrac{L^2}{8}\right), & L/4 \leqslant x \leqslant L \end{cases} \tag{30}$$

$$M^-(x) = \min_t M^e(x,t) = \begin{cases} -q_0^+\left(\dfrac{x^2}{2} - \dfrac{5Lx}{8} + \dfrac{L^2}{8}\right), & 0 \leqslant x \leqslant L/4 \\ -q_0^-\left(\dfrac{x^2}{2} - \dfrac{5Lx}{8} + \dfrac{L^2}{8}\right), & L/4 \leqslant x \leqslant L \end{cases}. \tag{31}$$

Application of (24)–(26) yields

$$\begin{aligned} A &= \max_{0\leqslant x\leqslant L} \frac{M^+(x) - M^-(x)}{M_Y^+ + M_Y^-} \\ &= \max_{0\leqslant x\leqslant L} \frac{q_0^+ - q_0^-}{M_Y^+ + M_Y^-}\left|\frac{x^2}{2} - \frac{5Lx}{8} + \frac{L^2}{8}\right| = \frac{L^2(q_0^+ - q_0^-)}{8(M_Y^+ + M_Y^-)} \end{aligned} \tag{32}$$

(here the maximum is reached at the point $x = 0$—that is the point of potential alternating plasticity collapse),

$$I = \sup_{\Delta\kappa\in\mathscr{C}} \frac{\displaystyle\int_0^L \max\{M^+(x)\Delta\kappa, M^-(x)\Delta\kappa\}\,\mathrm{d}x}{\displaystyle\int_0^L D(\Delta\kappa)\,\mathrm{d}x}. \tag{33}$$

To evaluate I we take an admissible incremental mechanism $\Delta\kappa$ with plastic hinges at $x = 0$ and $x = x_0$ [see Fig. 1, the free variable x_0 then should be chosen to maximize I in eqn (33)]:

$$\Delta\kappa = \theta_0 \cdot \delta(x) + \theta_H \cdot \delta(x - x_0). \tag{34}$$

At small deflections of the beam, the angles θ_0, θ_H can be given as

$$\theta_0 = -\frac{w_0}{x_0}\,(w_0 \text{ is the deflection}),$$

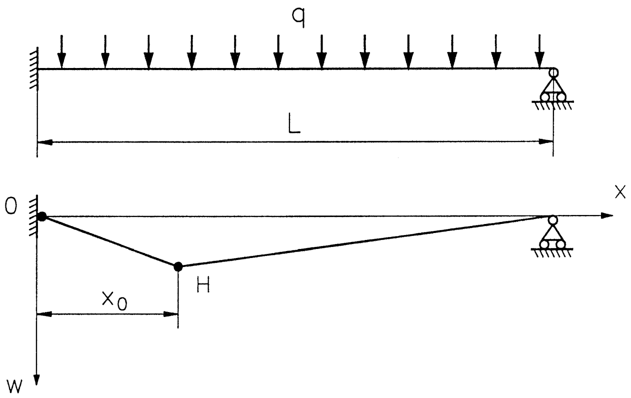

Fig. 1. A beam under uniform loads.

$$\theta_H = \frac{w_0}{x_0} + \frac{w_0}{L-x_0} = -\theta_0\left(1+\frac{x_0}{L-x_0}\right). \tag{35}$$

Substituting eqns (30), (31), (34) and (35), into (33), one will get

$$I = \max\{I_1, I_2\}, \tag{36}$$

where

$$I_1 = \sup_{0<x_0<L} \frac{q_0^+\left[\frac{L^2}{8} - \left(\frac{x_0^2}{2} - \frac{5Lx_0}{8} + \frac{L^2}{8}\right)\left(1+\frac{x_0}{L-x_0}\right)\right]}{M_Y^- + M_Y^+\left(1+\frac{x_0}{L-x_0}\right)},$$

$$I_2 = \sup_{0<x_0<L} \frac{-q_0^-\left[\frac{L^2}{8} - \left(\frac{x_0^2}{2} - \frac{5Lx_0}{8} + \frac{L^2}{8}\right)\left(1+\frac{x_0}{L-x_0}\right)\right]}{M_Y^+ + M_Y^-\left(1+\frac{x_0}{L-x_0}\right)}. \tag{37}$$

I_1 corresponds to the incremental collapse in the downward direction ($w_0 < 0$), while I_2 represents the upward mode ($w_0 < 0$). Formulae (24), (32), (36), (37) determined the shakedown limits q_0^+, q_0^- corresponding to $k_s = 1$. Though one of the external values of the elastic moment M^e is attained at $x = 3L/8$ [the other one is at $x = 0$, consult eqns (30), (31)], the optimal point x_0 in eqns (37), which determines a plastic hinge for the most dangerous collapse mechanism, may not be the same (see the numerical illustration that followed).

More generally, we consider the same structure under quasi-periodic dynamic loading

$$q(t) = q_0 + q_1 \sin\omega t, \tag{38}$$

where q_0, q_1, ω are arbitrary quasi-static functions of time, which are confined to

$$q_0^- \leqslant q_0(t) \leqslant q_0^+, \quad 0 \leqslant q_1(t) \leqslant q_1^+, \quad 0 \leqslant \omega(t) \leqslant \omega_1. \tag{39}$$

Denote

$$\alpha = \left(\frac{m\omega^2}{EJ}\right)^{1/4}, \quad \alpha_1 = \left(\frac{m\omega_1^2}{EJ}\right)^{1/4}, \tag{40}$$

where m is the mass density, EJ—the bending stiffness of the beam.

The elastic moment response of the beam to the dynamic load (38) is much more complicated in comparison with that in the quasi-static case:

$$M^e(x,t) = -q_0\left(\frac{x^2}{2} - \frac{5Lx}{8} + \frac{L^2}{8}\right) + \frac{q_1}{2\alpha^2}\sin\omega t\left\{2\cos\alpha x - \frac{1}{\mathrm{ch}\alpha L}(\cos\alpha x + \mathrm{ch}\alpha x)\right.$$
$$\left. + [(\sin\alpha x + \mathrm{sh}\alpha x) - (\cos\alpha x + \mathrm{ch}\alpha x)\mathrm{th}\alpha L]\frac{2\cos\alpha L\,\mathrm{ch}\alpha L - \mathrm{ch}\alpha L - \cos\alpha L}{\mathrm{sh}\alpha L\cos\alpha L - \mathrm{ch}\alpha L\sin\alpha L}\right\}. \tag{41}$$

Following the same steps as in the quasi-static case we evaluate $M^+(x)$, $M^-(x)$ from eqns (39), (41), and then A, I and k_s. We can take an admissible incremental mechanism as that of (34), (35) with x_0 being chosen to maximize I for a particular problem considered. In particular, substituting eqns (34), (35) into eqns (32), (33), one gets

$$k_s^{-1} = \max\{I, A\}, \tag{42}$$

where

$$A = (M_Y^+ + M_Y^-)^{-1} \cdot \max_{0 \leqslant x \leqslant L}[M^+(x) - M^-(x)],$$

$$I = \sup_{0 < x_0 < L} I(x_0), \quad I(x_0) = \max\{I_1(x_0), I_2(x_0)\},$$

$$I_1(x_0) = \left[M_Y^- + M_Y^+\left(1 + \frac{x_0}{L-x_0}\right)\right]^{-1}\left[-M^-(0) + M^+(x_0)\left(1 + \frac{x_0}{L-x_0}\right)\right],$$

$$I_2(x_0) = \left[M_Y^+ + M_Y^-\left(1 + \frac{x_0}{L-x_0}\right)\right]^{-1}\left[M^+(0) - M^-(x_0)\left(1 + \frac{x_0}{L-x_0}\right)\right].$$

For illustration, take $M_Y^+ = 4M_Y^- = 4M_Y$, $q_0^- = 0$. The shakedown curves $k_s = 1$, under which the structure is safe, in the plane of dimensionless load amplitude coordinates $\bar{q}_0 = q_0^+ L^2(35.88M_Y)$, $\bar{q}_1 = q_1^+ L^2/(35.88M_Y)$ at various values of dimensionless frequency bound $\bar{\alpha} = \alpha_1 L$ are presented in Fig. 2 ($\bar{q}_0 = 1$ is the unshakedown limit in the case of quasistatic loading $\bar{q}_1 \equiv 0$).

From Fig. 2 one sees that as $\bar{\alpha}$ and $\bar{q}_1$ increase, the limit $\bar{q}_0$ decreases drastically from the value $\bar{q}_0 = 1$ corresponding to the quasistatic case ($\bar{q}_1 \equiv 0$) toward $\bar{q}_0 = 0.244$ at $\bar{\alpha} = 2.85$ and $\bar{q}_1 = 0.04$, though $\bar{q}_1$ is relatively small compared with $\bar{q}_0$. Thus, the dynamic effect is strong. The calculations also indicate that, even in the quasistatic case, the optimal point $x_0 = 0.528L$ determining a plastic hinge of a potential mechanism does not coincide with the point $x = 0.625L$ where the elastic

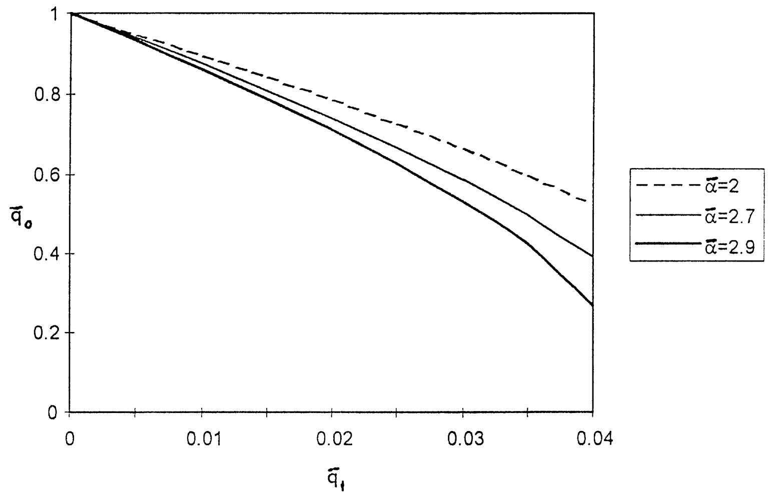

Fig. 2. The shakedown curves in the plane of load amplitudes (uniform loading).

moment M^e reaches its extremal value. As $\bar{\alpha}$ and $\bar{q}_1$ increase, the optimal point x_0 moves toward the right end $x = L$ (in particular, $x_0 = 0.698L$ at $\bar{\alpha} = 2$, $\bar{q}_1 = 0.04$, while $x_0 = 0.778L$ at $\bar{\alpha} = 2.7$, $\bar{q}_1 = 0.04$, and $x_0 = 0.872L$ at $\bar{\alpha} = 2.9$, $\bar{a}_1 = 0.04$). Thus the dynamic effect can change greatly the picture of the most dangerous collapse mechanism. Note that for the quasistatic loading, the three-point mechanism in Fig. 1 is instantaneous and is considered here as a trivial case of the more general incremental collapse mode, in which the deformation may increase step by step following load cycles. However for quasiperiodic dynamic loading, because of the inertia effect, the three-point mechanism appears incremental in the strict sense.

Next, we consider the same beam subjected to a quasi-periodic dynamic point load $P(t)$ at the point $x = x_p$ (Fig. 3)

$$P(t) = P_0 + P_1 \sin \omega t, \tag{43}$$

where P_0, P_1, ω are arbitrary quasi-static functions of time, which are confined to

$$P_0^- \leqslant P_0(t) \leqslant P_0^+, \quad 0 \leqslant P_1(t) \leqslant P_1^+, \quad 0 \leqslant \omega(t) \leqslant \omega_1. \tag{44}$$

With the notation (40), the elastic moment response of the beam to the load (43) has the form

$$M_e(x,t) = \begin{cases} \dfrac{P_1 \sin(\alpha(x_p - L)) \sin \omega t}{2\alpha(\cos \alpha L \operatorname{th}\alpha L - \sin \alpha L)}[-\sin \alpha x - \operatorname{sh}\alpha x + \operatorname{th}\alpha L(\cos \alpha x + \operatorname{ch}\alpha x)] \\ \quad + P_0 x\left(1 - \dfrac{3x_p^2}{2L^2} + \dfrac{x_p^3}{2L^3}\right) + P_0\left(-x_p + \dfrac{3x_p^2}{2L} + \dfrac{x_p^3}{2L^2}\right), \quad 0 \leqslant x \leqslant x_p \\ \dfrac{p_1 \sin \omega t}{2\alpha(\cos \alpha L \operatorname{th}\alpha L - \sin \alpha L)}\left[-\dfrac{\sin(\alpha(x_p - L))}{\operatorname{ch}\alpha L}\operatorname{sh}(\alpha(x - L)) - (\operatorname{th}\alpha L \cos \alpha x_p \right. \\ \quad \left. - \sin \alpha x_p) \sin(\alpha(x - L))\right] + P_0(L - x)\left(\dfrac{3x_p^2}{2L^2} - \dfrac{x_p^3}{2L^3}\right), \quad x_p \leqslant x \leqslant L \end{cases} \tag{45}$$

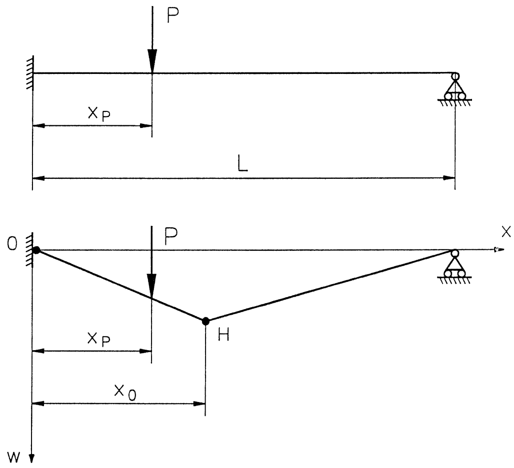

Fig. 3. A beam under a point load.

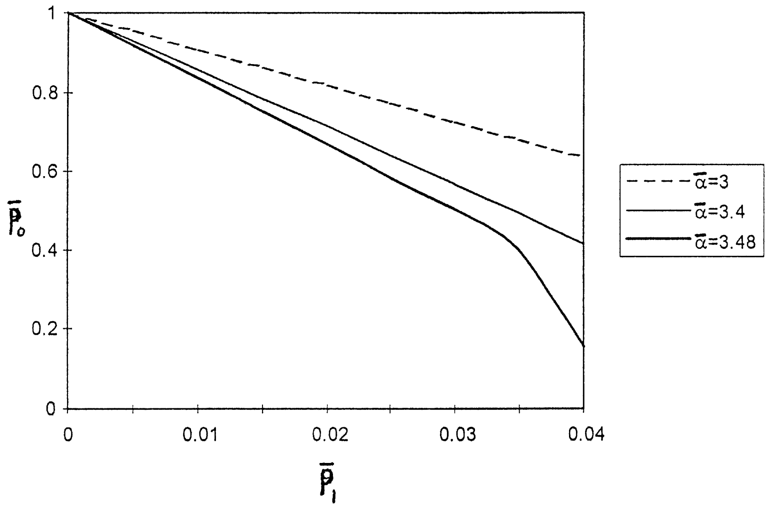

Fig. 4. The shakedown curves in the plane of load amplitudes (point loading).

The formulae (42) of the previous example apply there as well with the only difference in the particular expressions of the limits $M^+(x)$ and $M^-(x)$, which are determined from eqns (44), (45).

For illustration, take $M_Y^+ = 4M_Y^- = 4M_Y$, $P_0^- = 0$, $x_p = 0.5L$. The shakedown curves $k_s = 1$, under which the structure is safe, in the plane of dimensionless load amplitude coordinates $\bar{P}_0 = P_0^+L^2/(18M_Y)$, $\bar{P}_1 = P_1^+L^2/(18M_Y)$ at various values of dimensionless frequency bound $\bar{\alpha} = \alpha_1 L$ are presented in Fig. 4 ($\bar{P}_0 = 1$ is the unshakedown limited in the case of quasistatic loading $\bar{P}_1 \equiv 0$).

The same general tendency as that of the previous example is observed here, which indicates strong effects of the dynamic fluctuating part. Though in the quasistatic case the most dangerous collapse mechanism is the one with the plastic hinge at $x_0 = x_p$—the load point, in the dynamic case, this trivial observation may not be true. Numerical results indicate that at the bound $\bar{\alpha} = 3.4$ and $\bar{q}_1 = 0.04$, the critical x_0 is about $0.61L$ (recall that in our example $x_p = 0.5L$), while at $\bar{\alpha} = 3.48$ and $\bar{q}_1 = 0.04$, one get $x_0 = 0.88L$.

4. Conclusion

Shakedown analysis of planar beam structures, with generally different reinforcements in the upper and lower layers of the beams is given. An usual assumption (2) on elastic-perfectly plastic behaviour of the beam in bending is taken, so that the classical shakedown theory can apply. The reduced expression for the shakedown factor (24)–(26) has been constructed, which is equivalent to but simpler than the original formulation (4), (5), hence can be safely recommended for direct practical use without referring to the latter. The practical significance of the shakedown design in comparison with the more frequently used plastic limit design is that the former is safer and applies to a larger class of problems for structures under dynamic loading (Pham, 1992, 1996), which lie outside the framework of limit design. Shakedown analysis requires (generally-dynamic) elastic response of the structure to external agencies to be determined a priori, in particular its boundary $M_i^+(x_i)$ and $M_i^-(x_i)$ from (1). It might not be an easy task for general dynamic loading. However many dynamic loading processes one encounters in practice can be approximated by quasi-periodic ones, which are relatively easy for description as those in case of quasi-static loading. Strong impulsive loading processes can also be incorporated into consideration, once the respective elastic response of a structure (in particular, the limits of it in the stress space) has been determined a priori.

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